

Certain Generating Subspaces for Vertex Operator Algebras

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Minimal generating subspaces of “weak PBW type” for vertex operator algebras are studied and a procedure is developed for finding such subspaces. As applications, some results on generalized modules are obtained for vertex operator algebras that satisfy a certain condition, and a minimal generating space of weak PBW type is produced for V_L with L any positive-definite even lattice. © 1999

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1. INTRODUCTION

Vertex operator algebras are intimately related to Lie algebras, Lie groups, finite groups, and lattices. In particular, vertex operator algebras can be constructed from positive-definite even lattices [B, FLM], or from certain Lie algebras such as the Virasoro algebra and affine algebras [FZ]. (See also [DLe, DLM4, FKRW, Hua1, La, Li2, Lia, MP, P].) For instance, for an affine algebra $\hat{\mathfrak{g}}$ associated to a (finite-dimensional) simple Lie algebra \mathfrak{g} and for any complex number $\ell \neq -h$ with h being the dual Coxeter number, the generalized Verma module (or Weyl module) $V(\ell, 0)$ and its irreducible quotient $L(\ell, 0)$ each carries a natural vertex operator algebra structure.

For those vertex operator algebras constructed from Lie algebras, there are obvious generating subspaces. For example, \mathfrak{g} is a canonical generating space for both $V(\ell, 0)$ and $L(\ell, 0)$. Furthermore, if u^1, \dots, u^n is a basis of \mathfrak{g} and “ \leq ” is any linear order on the set $\{u_{-k}^i : k \geq 1, 1 \leq i \leq n\}$, then it follows from the Poincaré–Birkhoff–Witt (or PBW) theorem that

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$V(\ell, 0)$ has a basis

$$u_{-n_1}^{i_1} \cdots u_{-n_k}^{i_k} \mathbf{1}, \tag{1.1}$$

where $u_{-n_1}^{i_1} \leq \cdots \leq u_{-n_k}^{i_k}$ and $n_1, \dots, n_k \geq 1$. Note, however, for the vertex operator algebras $L(\ell, 0)$, the elements (1.1) may be linearly dependent.

It is of fundamental importance to study the structure of vertex operator algebras in terms of generators and defining relations. This is part of our motivation for this paper. Here, for an abstract vertex operator algebra, we study generating spaces U which play a role analogous to that of \mathfrak{g} in $L(\ell, 0)$.

We consider vertex operator algebras V such that $V = \bigoplus_{n=0}^\infty V_{(n)}$ and $V_{(0)} = \mathbb{C}\mathbf{1}$. For such a vertex operator algebra V , set $V_+ = \bigoplus_{n=1}^\infty V_{(n)}$, and define $C_1(V)$ to be the subspace of V linearly spanned by $u_{-1}v$ for $u, v \in V_+$ and by $L(-1)V$. (This is similar to the notion of Zhu's C_2 subspace $[Z]$, where C_2 is the linear span of $u_{-2}v$ for $u, v \in V$.) Let U be a graded subspace of V_+ such that $V_+ = U + C_1(V)$. It was proved in [Li3] that V as a vertex operator algebra is generated by U in the sense that V is generated from the vacuum vector by repeatedly applying u_m for $u \in U$, $m \in \mathbb{Z}$. We refine this result by proving that U plays the same role here as \mathfrak{g} plays for $L(\ell, 0)$ (without the linear independence). For convenience we call U a *generating space of weak PBW type*. It is also proved that if $V_+ = U \oplus C_1(V)$, then U is minimal among such generating spaces. Then calculating $C_1(V)$ gives us a way of finding a minimal generating space of weak PBW type.

In classical Lie theory, the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ together with the PBW theorem plays a crucial role. Here we are naturally led to the universal enveloping algebra $\mathcal{U}(V)$ associated to V , introduced by Frenkel and Zhu in [FZ]. By definition, $\mathcal{U}(V)$ is a topological associative algebra generated by symbols $v(m)$ (linear in v) for $v \in V$, $m \in \mathbb{Z}$ subject to relations corresponding to the defining identities for vertex operator algebras: the vacuum property, the $L(-1)$ -derivative property, and the Jacobi identity. Then the category of weak V -modules (Definition 2.1) is naturally equivalent to the category of restricted continuous $\mathcal{U}(V)$ -modules (defined in Section 2). Let u^i run through a basis of U consisting of homogeneous elements. Define a linear order “ \leq ” on $\{u^i(m) | i \geq 1, m \in \mathbb{Z}\}$ by defining $u^i(m) < u^j(n)$ if either $\text{wt } u^i - m - 1 > \text{wt } u^j - n - 1$ or $\text{wt } u^i - m - 1 = \text{wt } u^j - n - 1$ with $i > j$. With the obvious notion of nonincreasing monomial in the $u^i(n)$'s we prove that such monomials span a dense subspace of $\mathcal{U}(V)$. As corollaries we have: (a) V is spanned by nonincreasing monomials in the $u^i(-n)$'s for $n \geq 1$ applied to the vacuum vector $\mathbf{1}$; and (b) any lowest weight generalized V -module $W = \bigoplus_{n \in \mathbb{N}} W_{(h+n)}$

with lowest weight h is spanned by nonincreasing monomials in the $u^i(m)$'s, with $m < \text{wt } u^i - 1$, applied to $W_{(h)}$.

It is known [DLM2, Z] that Zhu's C_2 -finiteness condition implies that Zhu's algebra $A(V)$ (cf. [Z]) is finite-dimensional, so it follows from Zhu's one-to-one correspondence (cf. [Z]) that V has only finitely many irreducible lowest weight generalized modules up to equivalence. From our results Zhu's C_2 -finiteness condition further implies that each irreducible lowest weight generalized module is a module because Zhu's C_2 -finiteness condition implies that $C_1(V)$ is finite-codimensional (Proposition 3.16). As a corollary, under Zhu's C_2 -finiteness condition on V , if a generalized V -module is finitely generated and lower-truncated, then it is an ordinary module. The same conclusion was obtained in [Hua2] under different hypotheses.

Among the vertex operator algebras known to satisfy Zhu's C_2 -finiteness condition (cf. [Z, DLM3]) are the moonshine module vertex operator algebra V^\natural and the vertex operator algebras V_L associated to positive-definite even lattices L , so our main result implies that V_L and V^\natural have finite-dimensional generating subspaces of weak PBW type. Here we produce such a minimal generating space of V_L . Let L be a positive-definite even lattice. Define

$$\Phi(L) = \{0 \neq \alpha \in L \mid \langle \alpha - \beta, \beta \rangle < 0 \text{ for every } \beta \neq 0, \alpha\}.$$

Let $\mathbf{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$. We prove that the subspace $U = \mathbf{h} + \sum_{\alpha \in \Phi(L)} \mathbb{C} \iota(e_\alpha)$ of V_L (see Section 5 for the notation) is a minimal generating space of weak PBW type. We also show that $\Phi(L)$ satisfies several properties of a root system and, in case L is a root lattice, that $\Phi(L) = L_2$, the set of elements of squared length 2 in L .

Let σ be an automorphism of finite order of V . We define a twisted universal enveloping algebra $\mathcal{U}_\sigma(V)$ (cf. [DLin]) and show that results parallel to those described above hold for σ -twisted generalized V -modules.

Some ideas from G. M. T. Watt's paper [Wat] on W -algebras play a crucial role in the proof of our main results in Section 3.

Set $Q(V) = V_+/C_1(V)$. Then $Q(V) = \bigoplus_{n=1}^{\infty} Q(V)_n$ is a graded vector space. From Corollary 3.13, the graded vector space $Q(V)$ "measures" the vertex operator algebra V in some sense. In ongoing research [KL] we study vertex operator algebras V with $Q(V)$ of certain special types, and we have proved the following: (1) If V is a vertex operator algebra such that $Q(V) = Q(V)_1$, then V is isomorphic to the vertex operator algebra associated to an affine Lie algebra $\hat{V}_{(1)}$ (cf. [FZ]); and (2) if V is a vertex operator algebra such that $Q(V) = Q(V)_2$, then $V_{(2)}$ has a natural Frobenius algebra structure and V is isomorphic to the vertex operator algebra

associated to $V_{(2)}$ by Lam in [La] (see also [DLM4, P]). In view of these results, we note that the present paper provides a new framework for classifying vertex operator algebras.

This paper is organized as follows: In Section 2 we recall $\mathcal{U}(V)$. In Section 3 we present our main results on $\mathcal{U}(V)$ and some of their consequences. In Section 4 we study the twisted analogue $\mathcal{U}_\sigma(V)$. In Section 5 we present a minimal generating space of weak PBW type for the vertex operator algebra V_L .

2. THE UNIVERSAL ENVELOPING ALGEBRA

In this paper we use standard notions as defined in [FLM] and [FHL]. Throughout this paper, V will be a fixed vertex operator algebra and for simplicity we assume that $V = \bigoplus_{n=0}^{\infty} V_{(n)}$ and $V_{(0)} = \mathbb{C}\mathbf{1}$. Denote by \mathbb{Z} the set of integers, by \mathbb{N} the set of nonnegative integers, and by \mathbb{C} the field of complex numbers.

We shall also need the following notions.

DEFINITION 2.1. A *generalized V -module* [HL] satisfies all the axioms given in [FLM] and [FHL] for a V -module except for the two restrictions on the homogeneous subspaces. Namely, homogeneous subspaces need not be finite-dimensional, and the grading need not be lower-truncated. A generalized V -module W is called a *lowest weight generalized V -module* if $W = \bigoplus_{n \in \mathbb{N}} W_{(h+n)}$ for some $h \in \mathbb{C}$ and W as a V -module is generated by $W_{(h)}$, where $W_{(\lambda)}$ is the $L(0)$ -eigenspace of eigenvalue λ for $\lambda \in \mathbb{C}$. A *weak V -module* [DLM1] satisfies all the axioms for a V -module except for the axioms involving the $L(0)$ -grading. That is, there is no grading assumed for a weak module. A weak V -module W is said to be *\mathbb{N} -gradable* (cf. [Z]) if there exists an \mathbb{N} -grading $W = \bigoplus_{n \in \mathbb{N}} W(n)$ such that

$$u_m W(n) \subseteq W(\text{wt } u - m - 1 + n) \quad (2.1)$$

for homogeneous $u \in V$ and for $m \in \mathbb{Z}$, $n \in \mathbb{N}$, where by definition $W(k) = 0$ for $k < 0$.

Among the axioms in the definition of a vertex operator algebra V , the most important one is the Jacobi identity

$$\begin{aligned} z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) Y(u, z_1) Y(v, z_2) - z_0^{-1} \delta \left(\frac{z_2 - z_1}{-z_0} \right) Y(v, z_2) Y(u, z_1) \\ = z_2^{-1} \delta \left(\frac{z_1 - z_0}{z_2} \right) Y(Y(u, z_0)v, z_2) \end{aligned} \quad (2.2)$$

for $u, v \in V$. For $l, m, n \in \mathbb{Z}$, by taking $\text{Res}_{z_0} \text{Res}_{z_1} \text{Res}_{z_2} z_0^l z_1^m z_2^n$ from (2.2) we obtain

$$\begin{aligned} \sum_{i \geq 0} (-1)^i \binom{l}{i} u_{m+l-i} v_{n+i} - \sum_{i \geq 0} (-1)^{l+i} \binom{l}{i} v_{n+l-i} u_{m+i} \\ = \sum_{i \geq 0} \binom{m}{i} (u_{l+i} v)_{m+n-i}. \end{aligned} \quad (2.3)$$

In particular, by setting $l = 0$ and $m = 0$, respectively, we get [B, FLM]

$$[u_m, v_n] = \sum_{i=0}^{\infty} \binom{m}{i} (u_i v)_{m+n-i}, \quad (2.4)$$

$$(u_l v)_n = \sum_{i=0}^{\infty} \binom{l}{i} ((-1)^i u_{l-i} v_{n+i} - (-1)^{l+i} v_{l+n-i} u_i). \quad (2.5)$$

Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be a \mathbb{Z} -graded associative algebra with identity. We are going to define a certain topology on A (cf. [FZ]). One could equally well suppose that A is graded by $\frac{1}{k}\mathbb{Z}$ for a positive integer k . For each homogeneous subspace A_n , define a subspace $A_{n,k}$ for each $k \in \mathbb{Z}$ by

$$A_{n,k} = \sum_{i \geq 0} A_{n+k+i} A_{-k-i}. \quad (2.6)$$

Endow each homogeneous subspace A_n with the topology generated by the subspaces $\{A_{n,k}, k \geq 0\}$ and their translates. The topologies on all subspaces A_n induce a topology on the whole space A . Clearly,

$$A_{n,k} A_{m,l} \subseteq A_{m+n,l}, \quad (2.7)$$

and, therefore, the multiplication on A is continuous.

In general, the topology on A may fail to be separated, as shown by the Laurent polynomial algebra $\mathbb{C}[t, t^{-1}]$ graded by degree. By contrast, if A is a free graded associative algebra, then $\bigcap_{k \geq 0} A_{n,k} = \{0\}$ and $\bigcup_{k \geq 0} A_{n,-k} = A_n$ for every $n \in \mathbb{Z}$. For $n \in \mathbb{Z}$, let \bar{A}_n be the completion of A_n . The direct sum $\bar{A} = \bigoplus_{n \in \mathbb{Z}} \bar{A}_n$ is a \mathbb{Z} -graded topological algebra with unit element. When A is free, A is embedded in \bar{A} because each A_n is separated.

Let U be a \mathbb{Z} -graded vector space and let $L(U) = \mathbb{C}[t, t^{-1}] \otimes U$. Make $L(U)$ a \mathbb{Z} -graded vector space by defining

$$\deg(t^m \otimes u) = \deg u - m - 1 \quad (2.8)$$

for u a homogeneous vector in U and for $m \in \mathbb{Z}$. Let $T(L(U))$ be the tensor algebra of $L(U)$. Then $A = T(L(U))$ is a free \mathbb{Z} -graded algebra, and so carries the uniform structure defined above. The graded completion of $T(L(U))$, defined by $\bar{T}(L(U)) = \bigoplus_{n \in \mathbb{Z}} \bar{A}_n$, is a topological \mathbb{Z} -graded algebra and $T(L(U))$ is embedded in $\bar{T}(L(U))$. In particular, to every vertex operator algebra V , we associate a topological \mathbb{Z} -graded algebra $\bar{T}(L(V))$.

Denote the image of $t^m \otimes a$ in $\bar{T}(L(V))$ by $a(m)$. Then following [FZ], we define $\mathcal{U}(V)$ to be the quotient algebra of $\bar{T}(L(V))$ modulo the *Jacobi identity relations* (cf. (2.3))

$$\begin{aligned} & \sum_{i \geq 0} (-1)^i \binom{l}{i} u(m+l-i)v(n+i) \\ & - \sum_{i \geq 0} (-1)^{l+i} \binom{l}{i} v(n+l-i)u(m+i) \\ & = \sum_{i \geq 0} \binom{m}{i} (u_{l+i}v)(m+n-i) \end{aligned} \quad (2.9)$$

for $u, v \in V$ and $l, m, n \in \mathbb{Z}$; the $L(-1)$ -derivative identity relation

$$(L(-1)u)(m) = -mu(m-1) \quad (2.10)$$

for $u \in V$, $m \in \mathbb{Z}$; and the *vacuum identity relation*

$$\mathbf{1}(m) = \delta_{-1, m}. \quad (2.11)$$

From this tautological construction of $\mathcal{U}(V)$, it is clear that every weak V -module W is naturally a continuous $\mathcal{U}(V)$ -module, where $v(m)$ is represented by v_m for $v \in V$, $m \in \mathbb{Z}$. By the definition of a weak module, for any $v \in V$, $w \in W$ we have $v_m w = 0$ for m sufficiently large. We say that a $\mathcal{U}(V)$ -module M is *restricted* if, for every $v \in V$, $w \in M$, $v(m)w = 0$ for m sufficiently large. Then it is clear that the category of continuous restricted $\mathcal{U}(V)$ -modules is equivalent to the category of weak V -modules. (As mentioned by Dong, in the above equivalence of categories it is necessary to consider continuous $\mathcal{U}(V)$ -modules.)

Let ψ be a vertex operator algebra homomorphism from V^1 to V^2 . Then it is easy to see that ψ induces a continuous algebra homomorphism $\mathcal{U}(\psi)$ from $\mathcal{U}(V^1)$ to $\mathcal{U}(V^2)$ such that

$$\mathcal{U}(\psi)(a(m)) = \psi(a)(m) \quad \text{for } a \in V^1, m \in \mathbb{Z}.$$

Then \mathcal{U} is a functor from the category of vertex operator algebras to the category of \mathbb{Z} -graded topological associative algebras.

Remark 2.2. Let $\epsilon: L(V) \rightarrow \mathcal{U}(V)$ be the natural linear map. Then we have the following universal property. Suppose that A is a \mathbb{Z} -graded topological associative algebra with unit element 1_A and $\phi: L(V) \rightarrow A$ is a grade-preserving linear map such that the following relations hold in A :

$$\begin{aligned} & \sum_{i \geq 0} (-1)^i \binom{l}{i} \phi(u(m+l-i)) \phi(v(n+i)) \\ & - \sum_{i \geq 0} (-1)^{l+i} \binom{l}{i} \phi(v(n+l-i)) \phi(u(m+i)) \\ & = \sum_{i \geq 0} \binom{m}{i} \phi((u_{l+i}v)(m+i-i)) \end{aligned} \quad (2.12)$$

for $u, v \in V$ and $l, m, n \in \mathbb{Z}$, where the convergence of the three infinite sums is also part of the assumption;

$$\phi((L(-1)u)(m)) = -m\phi(u(m-1)) \quad (2.13)$$

for $u \in V$, $m \in \mathbb{Z}$; and

$$\phi(\mathbf{1}(m)) = \delta_{-1, m} 1_A \quad (2.14)$$

for $m \in \mathbb{Z}$. Then there is a unique grade-preserving algebra homomorphism $\alpha: \mathcal{U}(V) \rightarrow A$ such that $\phi = \alpha \circ \epsilon$.

3. WEAK PBW-TYPE GENERATING SUBSPACES FOR $\mathcal{U}(V)$

This section is the core of the paper. We recall the definition of $C_1(V)$ from [Li3] and prove that every graded subspace U of V with $V_+ = U \oplus C_1(V)$ gives rise to a minimal generating space of $\mathcal{U}(V)$ with one property (spanning) of the two PBW properties (spanning and linear independence). We also present several consequences.

We first recall from [Li3] the definition of $C_1(V)$:

DEFINITION 3.1. Set $V_+ = \bigoplus_{n=1}^{\infty} V_{(n)}$. Define $C_1(V)$ to be the subspace of V linearly spanned by elements of type

$$u_{-1}v, \quad L(-1)w \quad (3.1)$$

for $u, v \in V_+$, $w \in V$.

It is clear that $C_1(V)$ is a graded subspace of V_+ . Since $u_{-r} = (1/(r-1)!) (L(-1)^{r-1}u)_{-1}$ for $u \in V$, $r \geq 1$, $u_{-r}v \in C_1(V)$ for $u, v \in V_+$, $r \geq 1$.

We recall Proposition 3.3 from [Li3]:

PROPOSITION 3.2. *Let U be a graded subspace of V_+ such that $V_+ = U + C_1(V)$. Then, as a vertex operator algebra, V is generated by U .*

Let U be a graded subspace of V_+ such that $V_+ = U + C_1(V)$. In the following we shall refine Proposition 3.2 by proving that V is linearly spanned by elements

$$u_{-k_1}^1 \cdots u_{-k_r}^r \mathbf{1} \quad (3.2)$$

for $r \in \mathbb{N}$, $u^1, \dots, u^r \in U$, and $k_1, \dots, k_r \geq 1$. (A further refinement will be given in Corollary 3.13.)

Let $P(U)$ be the subspace linearly spanned by elements (3.2). We define a filtration on $P(U)$ as follows: For $s \in \mathbb{N}$, let $P(U)_s$ be the subspace linearly spanned by elements of type (3.2) with $r \leq s$.

LEMMA 3.3. *Suppose that $V_{(0)} + \cdots + V_{(n)} \subseteq P(U)$ for some $n \in \mathbb{N}$. Let $u, v \in V_+$ be homogeneous vectors and $r \geq 1$ such that $u_{-r}v \in V_{(n+1)}$. Then $u_{-r}v \in P(U)$.*

Proof. Since $\text{wt } u + \text{wt } v + r - 1 = \text{wt}(u_{-r}v) = n + 1$ and $\text{wt } u, \text{wt } v > 0$, we have $\text{wt } u, \text{wt } v \leq n$, so that $u, v \in P(U)$ (by assumption).

We are going to use induction on $s \in \mathbb{N}$ to prove that $u_{-r}v \in P(U)$ for $u \in P(U)_s$, $r \geq 1$. For $s = 0, 1$, it is clear. Assume that it is true for s and let $u \in P(U)_{s+1}$. By the definition of $P(U)$, without loss of generality we may assume $u = u'_{-k}b$ where $u' \in U$, $b \in P(U)_s$, $k \geq 1$. By the iterate formula (2.5), we have

$$\begin{aligned} u_{-r}v &= (u'_{-k}b)_{-r}v \\ &= \sum_{j=0}^{\infty} \binom{-k}{j} \left((-1)^j u'_{-k-j} b_{-r+j} v - (-1)^{-k+j} b_{-k-r-j} u'_j v \right). \end{aligned} \quad (3.3)$$

Since $u' \in U \subseteq V_+$ and

$$\begin{aligned} n + 1 &= \text{wt}(u_{-r}v) = \text{wt}(u'_{-k-j} b_{-r+j} v) \\ &= (\text{wt } u' + k + j - 1) + \text{wt}(b_{-r+j} v), \end{aligned}$$

we have $\text{wt}(b_{-r+j}v) \leq n$, but then $b_{-r+j}v \in P(U)$ by assumption, so $u'_{-k-j}b_{-r+j}v \in P(U)$. Similarly, $u'_jv \in P(U)$. Since $b \in P(U)_s$, we have $b_{-k-r-j}u'_jv \in P(U)$ by the inductive assumption. This completes the induction. Thus $u_{-r}v \in P(U)$. ■

PROPOSITION 3.4. *Let U be a graded subspace of V_+ . If $V_+ = U + C_1(V)$, then the vertex operator algebra V is linearly spanned by elements of type (3.2), i.e., $V = P(U)$.*

Proof. We shall prove that $\sum_{j=0}^n V_{(j)} \subseteq P(U)$ by induction on n . It is clear for $n = 0$. Assume that $\sum_{j=0}^n V_{(j)} \subseteq P(U)$ for some $n \geq 0$. Let $w \in V_{(n+1)}$. Since $V_+ = U + C_1(V)$, we may write w as a sum of elements like $u_{-1}v$ for $u, v \in V_+$ and $L(-1)w'$ for $w' \in V_{(n)}$. By Lemma 3.3, we have $u_{-1}v \in P(U)$. By the inductive assumption we have $w' \in P(U)$. Since $[L(-1), u'_{-s}] = su'_{-s-1}$ for any $u' \in V$, $s \in \mathbb{Z}$, and $L(-1)\mathbf{1} = 0$, it is clear that $L(-1)P(U) \subseteq P(U)$. Thus $w = L(-1)w' \in P(U)$. Therefore $V_{(n+1)} \subseteq P(U)$. This concludes the proof. ■

In the following we prove that the converse of Proposition 3.4 also holds, so that we have:

THEOREM 3.5. *Let U be a graded subspace of V_+ . Then V is linearly spanned by*

$$u^1_{-k_1} \cdots u^r_{-k_r} \mathbf{1} \quad (3.4)$$

for $r \in \mathbb{N}$, $u^1, \dots, u^r \in U$, and $k_1, \dots, k_r \geq 1$ if and only if $V_+ = U + C_1(V)$.

Proof. We prove the converse of Proposition 3.4. Consider an element v of type (3.4). If $r \geq 2$, it is clear that v lies in $C_1(V)$. Suppose that $r = 1$. If $k_1 = 1$, we have $v = u^1_{-1} \mathbf{1} = u^1 \in U$. If $k_1 \geq 2$, by the $L(-1)$ -derivative property, we have

$$v = u^1_{-k_1} \mathbf{1} = \frac{1}{(k_1 - 1)!} L(-1)^{k_1-1} u^1 \in C_1(V).$$

The proposition follows. ■

Remark 3.6. It follows from Theorem 3.5 that any graded subspace U of V_+ with $V_+ = U \oplus C_1(V)$ is minimal among graded subspaces B of V_+ such that V is linearly spanned by elements $b^1_{-k_1} \cdots b^r_{-k_r} \mathbf{1}$ with the b^i 's in B and $k_1, \dots, k_r \geq 1$.

Remark 3.7. Let $V = L(\mathscr{L}, 0)$ be the vertex operator algebra [FZ, Li2] associated to an affine algebra $\hat{\mathfrak{g}}$ of level \mathscr{L} such that $-\mathscr{L}$ is not equal to the dual Coxeter number. Then it can be shown that \mathfrak{g} ($= V_{(1)}$) is the unique graded subspace complementary to $C_1(V)$. Similarly, if V is the vertex operator algebra associated to the Virasoro algebra of any central charge, then $\mathbb{C}\omega$ ($= V_{(2)}$) is the unique graded subspace complementary to $C_1(V)$.

From now on we assume that U is a graded subspace of V_+ such that $V = U + C_1(V)$, or, what is equivalent (by Theorem 3.5), V is linearly

spanned by elements in (3.4). We are going to prove that the subspace of $\mathcal{Z}(V)$ linearly spanned by all elements

$$u^{j_1}(n_1) \cdots u^{j_r}(n_r) \quad (3.5)$$

for $u^{j_1}, \dots, u^{j_r} \in U$, $n_1, \dots, n_r \in \mathbb{Z}$ is a dense subspace.

For $s \in \mathbb{N}$, we define B_s to be the subspace of $\mathcal{Z}(V)$ linearly spanned by all elements (3.5) with $\text{wt } u^{j_1} + \cdots + \text{wt } u^{j_r} \leq s$. By definition, $B_s \subseteq B_{s+1}$ for $s \in \mathbb{N}$. Let $\overline{B_s}$ be the closure of B_s . Then, for $s, t \in \mathbb{N}$,

$$B_s \cdot B_t \subseteq B_{s+t}, \quad B_s \cdot \overline{B_t} \subseteq \overline{B_{s+t}}, \quad \overline{B_s} \cdot \overline{B_t} \subseteq \overline{B_{s+t}}, \quad (3.6)$$

so, $\bigcup_{s=0}^{\infty} \overline{B_s}$ is a subalgebra of $\mathcal{Z}(V)$.

LEMMA 3.8. *For any homogeneous element v of V and for any $m \in \mathbb{Z}$,*

$$v(m) \in \overline{B}_{\text{wt } v}. \quad (3.7)$$

Proof. We are going to prove this by induction on $\text{wt } v$. If $\text{wt } v = 0$, it is clear because $V_{(0)} = \mathbb{C}\mathbf{1}$. Assume that, for some positive integer p , the lemma holds for every homogeneous $v \in V$ with $\text{wt } v \leq p$. Let $v \in V_{(p+1)}$. By Proposition 3.4, without loss of generality we may assume

$$v = u_{-k}v'$$

for (homogeneous) $u \in U$, $v' \in V$, and $k \geq 1$. Since $k \geq 1$,

$$\text{wt } v (= \text{wt}(u_{-k}v')) = (\text{wt } u + k - 1) + \text{wt } v' \geq \text{wt } u + \text{wt } v'; \quad (3.8)$$

hence it suffices to show that $v(m) \in \overline{B}_{\text{wt } u + \text{wt } v'}$. Notice that $\text{wt } v' < \text{wt } v = p + 1$ because $\text{wt } u \geq 1$, so, by the inductive assumption, $v'(n) \in \overline{B}_{\text{wt } v'}$ for $n \in \mathbb{Z}$. Obviously, $u(n) \in B_{\text{wt } u} \subseteq \overline{B}_{\text{wt } u}$ for $n \in \mathbb{Z}$. Then it follows from (3.6) that

$$(-1)^i u(-k-i)v'(m+i) - (-1)^{k+i} v'(-k+m-i)u(i) \in \overline{B}_{\text{wt } u + \text{wt } v'} \quad (3.9)$$

for $i \geq 0$. The Jacobi identity relations for $\mathcal{Z}(V)$ give, for every $m \in \mathbb{Z}$ (cf. (2.5)),

$$\begin{aligned} v(m) = \sum_{i \geq 0} \binom{-k}{i} & \left((-1)^i u(-k-i)v'(m+i) \right. \\ & \left. - (-1)^{k+1} v'(-k+m-i)u(i) \right); \end{aligned} \quad (3.10)$$

hence from (3.9) we have $v(m) \in \overline{B}_{\text{wt } u + \text{wt } v'} \subseteq \overline{B}_{\text{wt } v}$. This completes the induction, and so the proof. ■

By the definition of $\mathcal{U}(V)$, the subalgebra generated by $v(m)$ for $v \in V$, $m \in \mathbb{Z}$ is dense in $\mathcal{U}(V)$. Then as an immediate corollary of Lemma 3.8 we have:

PROPOSITION 3.9. *The subalgebra $\bigcup_{s \geq 0} \bar{B}_s$ is dense in $\mathcal{U}(V)$.*

Let U be a graded subspace of V_+ as before and let $\{u^1, u^2, \dots\}$ (possibly finite) be a basis for U . Let " \leq " be any linear order on the set

$$\{u^i(m) | i \geq 1, m \in \mathbb{Z}\}.$$

A monomial

$$X = u^{i_1}(n_1) \cdots u^{i_r}(n_r) \quad (3.11)$$

(an element of $\mathcal{U}(V)$) is said to be *nonincreasing* if

$$u^{i_1}(n_1) \geq \cdots \geq u^{i_r}(n_r). \quad (3.12)$$

Now we are ready to prove:

THEOREM 3.10. *Let U be a graded subspace of V_+ such that either one of the two equivalent conditions in Theorem 3.5 holds and let $\{u^1, u^2, \dots\}$ be a basis for U . Let " \leq " be any linear order on the set*

$$\{u^i(m) | i \geq 1, m \in \mathbb{Z}\}.$$

Then the subspace of $\mathcal{U}(V)$ spanned by all nonincreasing monomials is dense in $\mathcal{U}(V)$.

Proof. By Proposition 3.9, it suffices to prove that every monomial in the $u^i(m)$'s is a linear combination of nonincreasing monomials. This is similar to the proof of one part of the PBW theorem for the classical universal enveloping algebra of a Lie algebra (cf. [Di]). Since every monomial (3.11) can be written in the form $\sigma(X)$ for some nonincreasing monomial X and some permutation σ of the factors of X , by induction it will suffice to show that if $X \in B_s$, then

$$\sigma(X) - X \in \bar{B}_{s-1}.$$

Furthermore, it suffices to prove the last statement for σ a transposition of adjacent factors.

For $p, q \geq 1$, $m, n \in \mathbb{Z}$, we have

$$u^p(m)u^q(n) - u^q(n)u^p(m) = \sum_{i \geq 0} \binom{m}{i} (u_i^p u^q)(m+n-i) \quad (3.13)$$

(a finite sum in $\mathcal{Z}(V)$). Since, for $i \geq 0$,

$$\text{wt}(u_i^p u^q) = \text{wt } u^p + \text{wt } u^q - i - 1 \leq \text{wt } u^p + \text{wt } u^q - 1,$$

using Lemma 3.8 we have

$$(u_i^p u^q)(m + n - i) \in \bar{B}_{\text{wt } u^p + \text{wt } u^q - 1},$$

so that

$$u^p(m)u^q(n) - u^q(n)u^p(m) \in \bar{B}_{\text{wt } u^p + \text{wt } u^q - 1}.$$

Therefore $\sigma(X) - X \in \bar{B}_{s-1}$. The theorem follows. ■

Remark 3.11. In [Wat], Watts essentially proved that if there are homogeneous elements u^1, \dots, u^k of V such that V is linearly spanned by

$$u_{-n_1}^1 \cdots u_{-n_k}^k \mathbf{1} \quad (3.14)$$

for $n_1, \dots, n_k \geq 1$, then V can be spanned by elements (3.14) in the standard order (defined after this remark). We owe to the paper [Wat] the idea of defining a filtration, as above on B , and the idea for the proof of Theorem 3.10.

In order to study lowest weight V -modules, we shall use a certain “lexicographic” order on the set $\{u^i(m) | i \geq 1, m \in \mathbb{Z}\}$. We define $u^i(m) \geq u^j(n)$ if either

$$\deg(u^i(m)) \geq \deg(u^j(n)) \quad (3.15)$$

or

$$\deg(u^i(m)) = \deg(u^j(n)) \quad \text{and} \quad i \leq j. \quad (3.16)$$

A monomial $X = u^{i_1}(m_1) \cdots u^{i_r}(m_r)$ nonincreasing with respect to the order just defined above is said to be *standard*.

Let $W = \bigoplus_{n \in \mathbb{N}} W_{(h+n)}$ be a lowest weight generalized V -module with lowest weight subspace $W_{(h)}$. Then $W = \mathcal{Z}(V)W_{(h)}$. As an immediate consequence of Theorem 3.10 we have:

COROLLARY 3.12. *Any lowest weight generalized V -module $W = \bigoplus_{n \in \mathbb{N}} W_{(h+n)}$ is linearly spanned by elements*

$$u^{i_1}(n_1) \cdots u^{i_r}(n_r) w \quad \left(= u_{n_1}^{i_1} \cdots u_{n_r}^{i_r} w \right) \quad (3.17)$$

for standard monomials $u^{i_1}(n_1) \cdots u^{i_r}(n_r)$ with $\deg(u^{i_j}(n_j)) > 0$ and for $w \in W_{(h)}$.

Since $V = \bigoplus_{n \in \mathbb{N}} V_{(n)}$ and $V_{(0)} = \mathbb{C}\mathbf{1}$, we can specialize w to $\mathbf{1}$:

COROLLARY 3.13. *V is linearly spanned by elements $u_{n_1}^{i_1} \cdots u_{n_r}^{i_r} \mathbf{1}$ associated to standard monomials $u^{i_1}(n_1) \cdots u^{i_r}(n_r)$ with $\deg(u^{i_j}(n_j)) > 0$.*

As an immediate consequence of Corollary 3.12 we have:

COROLLARY 3.14. *Let V be such that $\dim V/C_1(V) < \infty$ and let $W = \bigoplus_{n=0}^{\infty} W_{(h+n)}$ be a lowest weight generalized V -module such that $\dim W_{(h)} < \infty$. Then all homogeneous subspaces of W are finite-dimensional; that is, W is an (ordinary) module.*

Remark 3.15. Let $W = \bigoplus_{n=0}^{\infty} W_{(n+h)}$ be a lowest weight V -module and set

$$\chi_W(\mathbf{1}, q) = \sum_{n=0}^{\infty} \dim W_{(n+h)} q^n, \quad (3.18)$$

where q is a formal variable for now. Then under the assumption of Corollary 3.14, it follows directly from Corollary 3.12 and the convergence of the η -function that $\chi_W(\mathbf{1}, q)$ converges absolutely for a complex number q with $|q| < 1$.

In [Z], Zhu constructed an associative algebra $A(V)$ and established a one-to-one correspondence between the set of equivalence classes of irreducible $A(V)$ -modules and the set of equivalence classes of irreducible generalized V -modules. Namely, for any irreducible generalized V -module W , the lowest weight subspace $W_{(h)}$ is a natural irreducible $A(V)$ -module; conversely, for any irreducible $A(V)$ -module E , there is an irreducible generalized V -module W such that the lowest weight subspace $W_{(h)}$ as an $A(V)$ -module is isomorphic to E . Also in [Z], Zhu introduced what he called the C_2 -finiteness condition, which we now recall.

Let $C_2(V)$ be the subspace of V , linearly spanned by $u_{-2}v$ for $u, v \in V$. The vertex operator algebra V is said to satisfy the C_2 -finiteness condition if $V/C_2(V)$ is finite-dimensional. A very important consequence of V being C_2 -finite, proved by Zhu in [Z], is that the “ q -trace-function” of V on any module satisfies a differential equation of a certain type. Then the convergence of the q -trace functions easily follows. As indicated in one of Zhu’s propositions in [Z] and explicitly proved in [DLM3] (Proposition 3.6), Zhu’s C_2 -finiteness condition implies that $A(V)$ is finite-dimensional; hence there are only finitely many irreducible generalized V -modules up to equivalence and so that the lowest weight subspace of any irreducible generalized V -module is finite-dimensional. Here we shall use our result to prove that Zhu’s C_2 -finiteness condition implies that all homogeneous subspaces of an irreducible generalized V -module are finite-dimensional,

so that any irreducible generalized V -module is an (ordinary) V -module. Note that the same conclusion was obtained in [DLM2] under the assumption that V is rational (without the C_2 -finiteness condition).

PROPOSITION 3.16. *We have*

$$C_2(V) \subseteq C_1(V). \quad (3.19)$$

In particular, Zhu's C_2 -finiteness condition implies that $C_1(V)$ is finite-codimensional.

Proof. Notice that $u_{-2}v = (L(-1)u)_{-1}v$ for $u, v \in V$. If $v \in V_+$, then $u_{-2}v \in C_1(V)$ because $u_{-2} = 0$ for $u \in V_{(0)} = \mathbb{C}\mathbf{1}$. If $v \in V_{(0)} = \mathbb{C}\mathbf{1}$, then $u_{-2}v = (L(-1)u)_{-1}v \in L(-1)V$. Therefore $C_2(V) \subseteq C_1(V)$. ■

Since Zhu's C_2 -finiteness condition on V implies $\dim A(V) < \infty$ [Z, DLM3], it follows from Zhu's one-to-one correspondence that V has only finitely many irreducible \mathbb{N} -gradable weak V -modules up to equivalence and that each of these has a finite-dimensional lowest weight subspace. By Proposition 3.16 and Corollary 3.14 we have:

COROLLARY 3.17. *Suppose that V satisfies Zhu's C_2 -finiteness condition. Then there are only finitely many irreducible generalized V -modules up to equivalence and each of them has finite-dimensional homogeneous subspaces.*

As an immediate consequence of Corollary 3.17 we have:

COROLLARY 3.18. *Suppose that V satisfies Zhu's C_2 -finiteness condition. Then any finitely generated and lower-truncated generalized V -module is a module.*

In [Hua2], Huang defined the notion of condition A (on both the algebra and the modules), and under this condition A proved that every finitely generated lower-truncated generalized V -module is a module. Parts (1) and (2) of Huang's condition A are related to the conclusion of Theorem 3.10; part (3) follows from Zhu's C_2 -finiteness condition.

Remark 3.19. In [Z], Zhu's main results hold under the assumption that V is rational and C_2 -finite. Actually, from the rationality assumption Zhu only used the consequences that $A(V)$ is semisimple and that every irreducible lowest weight generalized V -module has finite-dimensional homogeneous subspaces. By Corollary 3.18, Zhu's results in [Z] will hold if V satisfies the C_2 -finiteness condition and $A(V)$ is semisimple. Practically, it is easier to check the semisimplicity of $A(V)$ than the rationality of V .

Remark 3.20. In [Li3], it was proved that if $L(0)$ acts semisimply on every weak V -module, then V satisfies Zhu's C_2 -finiteness condition. On the other hand, it was proved in [DLM3] (cf. [Z]) that each of the following

vertex operator algebras satisfies Zhu's finiteness condition: (i) V_L associated to a positive-definite even lattice L ; (ii) $L_{\mathfrak{g}}(\ell, 0)$ associated to a finite-dimensional simple Lie algebra \mathfrak{g} and a positive integer ℓ ; (iii) $L(c_{p,q}, 0)$ associated to the Virasoro algebra with central charge $c_{p,q} = 1 - 6(p - q)^2/pq$, where p and q are relatively prime positive integers greater than 1; (iv) V^{\natural} , Frenkel, Lepowsky and Meurman's moonshine module; and (v) tensor products of any of the vertex operator algebras listed above.

It is well known (and obvious) that the vertex operator algebras $L(\ell, 0)$ (associated to an affine Lie algebra) and the vertex operator algebras $L(c, 0)$ (associated to the Virasoro algebra) have finite-dimensional generating subspaces of weak PBW type. However, we believe that the following is new:

COROLLARY 3.21. *All of the vertex operator algebras V_L associated to positive-definite even lattices L , as well as the moonshine module vertex operator algebra V^{\natural} , have finite-dimensional generating subspaces of weak PBW type.*

Remark 3.22. Notice that for the vertex operator algebras $L(\ell, 0)$ with a generic level ℓ , the standard monomials are linearly independent. On the other hand, the standard monomials are linearly dependent when ℓ is a positive integer. This is related to an interesting problem: finding a basis for all standard modules for an affine Lie algebra [LW].

Remark 3.23. Note that all results of this section hold if V is a quasi-vertex operator algebra [FHL] instead of a vertex operator algebra.

4. THE TWISTED ANALOGUE $\mathscr{U}_{\sigma}(V)$

Here we shall study the twisted universal enveloping algebra $\mathscr{U}_{\sigma}(V)$ associated to an automorphism σ of V of finite order and give some results parallel to those in Section 3.

Throughout this section, V will be a fixed vertex operator algebra V . An automorphism of V is a linear bijection σ from the underlying vector space V to itself such that

$$\sigma(\mathbf{1}) = \mathbf{1}, \quad (4.1)$$

$$\sigma(\omega) = \omega, \quad (4.2)$$

$$\sigma Y(u, z)v = Y(\sigma(u), z)\sigma(v) \quad (4.3)$$

for $u, v \in V$.

Let V be a fixed vertex operator algebra with an automorphism σ . Then $\sigma L(m) = L(m)\sigma$ for $m \in \mathbb{Z}$. Consequently, $\sigma V_{(n)} = V_{(n)}$ for $n \in \mathbb{Z}$.

Now suppose that the automorphism σ has finite order k . Then

$$V = V^0 \oplus \cdots \oplus V^{k-1}, \quad (4.4)$$

where

$$V^j = \{u \in V \mid \sigma(u) = e^{2j\pi\sqrt{-1}}u\}. \quad (4.5)$$

A σ -twisted weak V -module [Do2, FFR, FLM] is a vector space W , equipped with a linear map Y from V to $(\text{End } W)\{z\}$ (with rational powers of z) such that the following hold:

(T1) (the vacuum property) $Y(\mathbf{1}, z) = 1$;

(T2) (the $L(-1)$ -derivative property) $Y(L(-1)v, z) = \frac{d}{dz}Y(v, z)$ for $v \in V$;

(T3) (the truncation condition) $Y(u, z)w \in z^{-j/k}W((z))$ for $u \in V^j$, $w \in W$;

(T4) (the σ -twisted Jacobi identity)

$$\begin{aligned} & z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y(u, z_1)Y(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)Y(v, z_2)Y(u, z_1) \\ &= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\left(\frac{z_1 - z_0}{z_2}\right)^{-j/k}Y(Y(u, z_0)v, z_2) \end{aligned} \quad (4.6)$$

for $u \in V^j$, $v \in V$.

The notions of σ -twisted V -module, σ -twisted generalized V -module, and lowest weight generalized σ -twisted V -module are defined accordingly.

If $\alpha \in \mathbb{C}$, then from [FLM] we have

$$z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\left(\frac{z_1 - z_0}{z_2}\right)^{-\alpha} = z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)\left(\frac{z_2 + z_0}{z_1}\right)^{\alpha}. \quad (4.7)$$

For $l \in \mathbb{Z}$, $m, n \in \frac{1}{k}\mathbb{Z}$, by taking $\text{Res}_{z_0}\text{Res}_{z_1}\text{Res}_{z_2}z_0^l z_1^m z_2^n$ of (4.6) we obtain

$$\begin{aligned} & \sum_{i \geq 0} \binom{l}{i} ((-1)^i u_{l+m-i} v_{n+i} - (-1)^{l+i} v_{l+n-i} u_{m+i}) \\ &= \sum_{i \geq 0} \binom{m}{i} (u_{l+i} v)_{m+n-i}. \end{aligned} \quad (4.8)$$

Specializing l to 0, we get the *twisted commutator formula*

$$[u_m, v_n] = \sum_{i \geq 0} \binom{m}{i} (u_i v)_{m+n-i}, \quad (4.9)$$

where $m, n \in \frac{1}{k}\mathbb{Z}$.

Let $u \in V^j$ with $0 \leq j \leq k$ and $v \in V$. Then there is a nonnegative integer N such that

$$(z_1 - z_2)^N [Y(u, z_1), Y(v, z_2)] = 0 \quad (4.10)$$

(cf. [DLe]). Take $\text{Res}_{z_1} z_1^{j/k}$ of the σ -twisted Jacobi identity (4.6) in $\mathcal{U}_\sigma(V)$, first using (4.7) on the right-hand side, to get

$$\begin{aligned} & (z_2 + z_0)^{j/k} Y(Y(u, z_0)v, z_2) \\ &= \text{Res}_{z_1} z_1^{j/k} \left(z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) Y(u, z_1) Y(v, z_2) \right. \\ & \quad \left. - z_0^{-1} \delta \left(\frac{z_2 - z_1}{-z_0} \right) Y(v, z_2) Y(u, z_1) \right) \\ &= \text{Res}_{z_1} z_1^{j/k} \sum_{n < N} z_0^{-n-1} \left((z_1 - z_2)^n Y(u, z_1) Y(v, z_2) \right. \\ & \quad \left. - (-z_2 + z_1)^n Y(v, z_2) Y(u, z_1) \right). \quad (4.11) \end{aligned}$$

In view of (4.10) and (4.11), it is legitimate to write

$$\begin{aligned} & Y(Y(u, z_0)v, z_2) \\ &= (z_2 + z_0)^{-j/k} \text{Res}_{z_1} z_1^{j/k} \sum_{n < N} z_0^{-n-1} \left((z_1 - z_2)^n Y(u, z_1) Y(v, z_2) \right) \\ & \quad - (z_2 + z_0)^{-j/k} \text{Res}_{z_1} z_1^{j/k} \\ & \quad \times \sum_{n < N} z_0^{-n-1} \left((-z_2 + z_1)^n Y(v, z_2) Y(u, z_1) \right), \quad (4.12) \end{aligned}$$

since on the right-hand side, each of the formal Laurent series in z_0 is truncated from below. Note that without the restriction $n < N$ the products in (4.12) would not exist.

For $p \in \mathbb{Z}$ and $q \in \frac{1}{k}\mathbb{Z}$, by taking $\text{Res}_{z_0} \text{Res}_{z_2} z_0^p z_2^q$ of (4.12) we obtain the *twisted iterate formula*

$$\begin{aligned} (u_p v)_q &= \sum_{i \in \mathbb{N}, p \leq \mu < N} c(i, \mu) u_{j/k-i+\mu} v_{p+q-j/k+i-\mu} \\ & \quad - d(i, \mu) v_{p+q-j/k-i} u_{j/k+i}, \quad (4.13) \end{aligned}$$

where

$$c(i, \mu) = (-1)^i \binom{\frac{-j}{k}}{\mu - p} \binom{\mu}{i}$$

and

$$d(i, \mu) = (-1)^{i+\mu} \binom{\frac{-j}{k}}{\mu - p} \binom{\mu}{i}.$$

To define the σ -twisted universal enveloping algebra of V , set

$$\begin{aligned} L_\sigma(V) &= V^0 \otimes \mathbb{C}[t, t^{-1}] + V^1 \otimes t^{1/k} \mathbb{C}[t, t^{-1}] + \cdots \\ &\quad + V^{k-1} \otimes t^{(k-1)/k} \mathbb{C}[t, t^{-1}], \end{aligned} \quad (4.14)$$

and let $T(L_\sigma(V))$ be the tensor algebra. We call $u \in V$ homogeneous if $u \in V_{(n)} \cap V^j$ for some $n \in \mathbb{N}$ and integer j with $0 \leq j \leq k$. For homogeneous $u \in V$ and $n \in \frac{1}{k}\mathbb{Z}$, define

$$\deg(u \otimes t^n) = \text{wt } u - n - 1. \quad (4.15)$$

This makes $L_\sigma(V)$ into a $\frac{1}{k}\mathbb{Z}$ -graded vector space, and by the induced grading $T(L_\sigma(V))$ is $\frac{1}{k}\mathbb{Z}$ -graded. As in the case of a \mathbb{Z} -graded algebra, treated in Section 2, we define a topology on the algebra $A = T(L_\sigma(V))$, and let $\bar{T}(L_\sigma(V))$ be the graded completion $\oplus_{\lambda \in (1/k)\mathbb{Z}} \bar{A}_\lambda$. We define $\mathscr{U}_\sigma(V)$ to be the quotient algebra of $\bar{T}(L_\sigma(V))$ modulo relations parallel to (2.9)–(2.11): the vacuum identity relation, the $L(-1)$ -derivative identity relation, and the σ -twisted Jacobi identity relations in $\bar{T}(L_\sigma(V))$ corresponding to (4.8). Then the category of σ -twisted weak V -modules is naturally isomorphic to the category of restricted continuous $\mathscr{U}_\sigma(V)$ -modules.

For convenience, when there is no danger we abuse the notation to write $u(n) = u_n$ for $u \in V$, $n \in \frac{1}{k}\mathbb{Z}$, so that

$$Y(u, z) = \sum_{n \in (1/k)\mathbb{Z}} u_n z^{-n-1} = \sum_{n \in (1/k)\mathbb{Z}} u(n) z^{-n-1} \in \mathscr{U}_\sigma(V)\{z\}.$$

Then the σ -twisted Jacobi identity (4.6), commutator formula (4.9), and iterate formula (4.13) hold in $\mathscr{U}_\sigma(V)$.

We fix a graded vector space U of V_+ such that either one of the two equivalent conditions in Theorem 3.5 holds and, as before, we define the

notion of standard monomial by choosing an ordered basis of U . Then we have:

THEOREM 4.1. *The standard monomials span a dense subspace of $\mathcal{U}_\sigma(V)$.*

Proof. One first establishes the twisted analogue of Lemma 3.8. The proof of Lemma 3.8 goes through here if we use relation (4.13) in place of (3.10). Then the proof of Theorem 3.10 goes through here with minor changes. ■

As an immediate consequence of Theorem 4.1 we have:

COROLLARY 4.2. *Suppose that $V/C_1(V)$ is finite-dimensional. Let W be a lowest weight σ -twisted generalized V -module such that the lowest weight subspace is finite-dimensional. Then W is a σ -twisted (ordinary) V -module.*

In [DLM2], an associative algebra $A_\sigma(V)$ was associated to a vertex operator algebra V and an automorphism of finite order, σ of V , and a natural one-to-one correspondence was constructed between the set of equivalence classes of irreducible lowest weight generalized σ -twisted V -modules and the set of equivalence classes of irreducible $A_\sigma(V)$ -modules on which the central element ω acts as a scalar. It was proved in [DLM2] that the C_2 -finiteness condition on V implies that $A_\sigma(V)$ is finite-dimensional. Then we immediately have:

COROLLARY 4.3. *Suppose that $V/C_2(V)$ is finite-dimensional. Then V has only finitely many irreducible lowest weight generalized σ -twisted V -modules up to equivalence, and any lowest weight generalized σ -twisted V -module is a σ -twisted (ordinary) V -module.*

5. MINIMAL GENERATING SPACES FOR VERTEX OPERATOR ALGEBRAS V_L

Throughout this section L will be a fixed positive-definite even lattice.

We first recall from [FLM] the main ingredients in the construction of the vertex operator algebra V_L . Set $\mathbf{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend the \mathbb{Z} -form on L to \mathbf{h} . Let $\hat{\mathbf{h}} = \mathbb{C}[t, t^{-1}] \otimes \mathbf{h} \oplus \mathbb{C}c$ be the affinization of \mathbf{h} ; i.e., $\hat{\mathbf{h}}$ is a Lie algebra with commutator relations

$$[t^m \otimes h, t^n \otimes h'] = m \delta_{m+n, 0} \langle h, h' \rangle c \quad \text{for } h, h' \in \mathbf{h}, m, n \in \mathbb{Z}, \quad (5.1)$$

$$[\hat{\mathbf{h}}, c] = 0. \quad (5.2)$$

We also use the notation $h(n) = t^n \otimes h$ for $h \in \mathbf{h}, n \in \mathbb{Z}$.

Set

$$\hat{\mathbf{h}}^+ = t\mathbb{C}[t] \otimes \mathbf{h}, \quad \hat{\mathbf{h}}^- = t^{-1}\mathbb{C}[t^{-1}] \otimes \mathbf{h}. \quad (5.3)$$

Then $\hat{\mathbf{h}}^+$ and $\hat{\mathbf{h}}^-$ are abelian subalgebras of $\hat{\mathbf{h}}$. Let $U(\hat{\mathbf{h}}^-) = S(\hat{\mathbf{h}}^-)$ be the universal enveloping algebra of $\hat{\mathbf{h}}^-$. Consider the induced $\hat{\mathbf{h}}$ -module

$$M(1) = U(\hat{\mathbf{h}}) \otimes_{U(\mathbb{C}[t] \otimes \mathbf{h} \oplus \mathbb{C}c)} \mathbb{C} \simeq S(\hat{\mathbf{h}}^-) \quad (\text{linearly}), \quad (5.4)$$

where $\mathbb{C}[t] \otimes \mathbf{h}$ acts trivially on \mathbb{C} and c acts on \mathbb{C} as multiplication by 1.

Let \hat{L} be the canonical central extension of L by the cyclic group $\langle \pm 1 \rangle$

$$1 \rightarrow \langle \pm 1 \rangle \rightarrow \hat{L} \xrightarrow{\bar{}} L \rightarrow 1, \quad (5.5)$$

with the commutator map $c(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle}$ for $\alpha, \beta \in L$. Let $e: L \rightarrow \hat{L}$ be a section such that $e_0 = 1$ and let $\epsilon: L \times L \rightarrow \langle \pm 1 \rangle$ be the corresponding 2-cocycle; i.e., for $\alpha, \beta, \gamma \in L$, $e_\alpha e_\beta = \epsilon(\alpha, \beta)e_{\alpha+\beta}$. Then

$$\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{\langle \alpha, \beta \rangle}, \quad (5.6)$$

$$\epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma) = \epsilon(\beta, \gamma)\epsilon(\alpha, \beta + \gamma) \quad (5.7)$$

for $\alpha, \beta, \gamma \in L$.

Form the induced \hat{L} -module

$$\mathbb{C}\{L\} = \mathbb{C}[\hat{L}] \otimes_{\langle \pm 1 \rangle} \mathbb{C} \simeq \mathbb{C}[L] \quad (\text{linearly}), \quad (5.8)$$

where $\mathbb{C}[\cdot]$ denotes the group algebra and -1 acts on \mathbb{C} as multiplication by -1 . For $a \in \hat{L}$, write $\iota(a)$ for $a \otimes 1$ in $\mathbb{C}\{L\}$. Then the action of \hat{L} on $\mathbb{C}\{L\}$ is given by $a \cdot \iota(b) = \iota(ab)$ and $(-1) \cdot \iota(b) = -\iota(b)$ for $a, b \in \hat{L}$. Furthermore we define an action of \mathbf{h} on $\mathbb{C}\{L\}$ by $h \cdot \iota(a) = \langle h, \bar{a} \rangle \iota(a)$ for $h \in \mathbf{h}$, $a \in \hat{L}$, and define $z^h \cdot \iota(a) = z^{\langle h, \bar{a} \rangle} \iota(a)$.

The underlying vector space for the vertex operator algebra V_L is

$$V_L = \mathbb{C}\{L\} \otimes_{\mathbb{C}} M(1) \simeq \mathbb{C}[L] \otimes S(\hat{\mathbf{h}}^-) \quad (\text{linearly}), \quad (5.9)$$

and $\hat{L}, \hat{\mathbf{h}}, z^h$, $h \in \mathbf{h}$, act naturally on V_L by acting on either $\mathbb{C}\{L\}$ or $M(1)$ as indicated above.

By identifying $h \in \mathbf{h}$ with $1 \otimes h(-1) \in V_L$ we consider \mathbf{h} as a subspace of V_L . The vertex operator $Y(h, z)$ associated to $h \in \mathbf{h}$ is given by

$$Y(h, z) = \sum_{n \in \mathbb{Z}} h(n) z^{-n-1}. \quad (5.10)$$

That is, $h_n = h(n)$ for $n \in \mathbb{Z}$. For $a \in \hat{L}$, the vertex operator $Y(\iota(a), z)$ associated to $\iota(a)$ is

$$Y(\iota(a), z) = \exp\left(\sum_{n=1}^{\infty} \frac{\bar{a}(-n)}{n} z^n\right) \exp\left(\sum_{n=1}^{\infty} \frac{\bar{a}(n)}{-n} z^{-n}\right) a z^{\bar{a}}. \quad (5.11)$$

Let $\{h^i | i = 1, \dots, d\}$ be an orthonormal basis of \mathbf{h} . Then the Virasoro element of V_L is given by

$$\omega = \frac{1}{2} \sum_{i=1}^d h^i(-1) h^i(-1) \in V_L. \quad (5.12)$$

Clearly, V_L is linearly spanned by elements

$$v = \iota(a) \otimes h^{i_1}(-n_1) \cdots h^{i_k}(-n_k) \quad (5.13)$$

for $a \in \hat{L}$, $k \in \mathbb{N}$, $i_1 \leq \cdots \leq i_k$, $n_1, \dots, n_k \in \mathbb{Z}$ ($n_i > 0$). We have

$$\begin{aligned} L(0)(\iota(a) \otimes h^{i_1}(-n_1) \cdots h^{i_k}(-n_k)) \\ = \left(\frac{1}{2} \langle \bar{a}, \bar{a} \rangle + n_1 + \cdots + n_k\right) (\iota(a) \otimes h^{i_1}(-n_1) \cdots h^{i_k}(-n_k)) \end{aligned} \quad (5.14)$$

and

$$\begin{aligned} L(-1)(\iota(a) \otimes h^{i_1}(-n_1) \cdots h^{i_k}(-n_k)) \\ = \bar{a}(-1)v + n_1(\iota(a) \otimes h^{i_1}(-1-n_1)h^{i_2}(-n_2) \cdots h^{i_k}(-n_k)) + \cdots \\ + n_k(\iota(a) \otimes h^{i_1}(-n_1) \cdots h^{i_{k-1}}(-n_{k-1})h^{i_k}(-1-n_k)) \end{aligned} \quad (5.15)$$

(cf. [Do1]). Since L is positive-definite, we have $V_L = \bigoplus_{n=0}^{\infty} (V_L)_{(n)}$ and $(V_L)_{(0)} = \mathbb{C}\mathbf{1}$.

Suppose that $h \in \mathbf{h}$ and $a \in \hat{L}$, by (2.4). Then

$$[h(m), \iota(a)_n] = \langle \bar{a}, h \rangle \iota(a)_{m+n} \quad (5.16)$$

for every $m, n \in \mathbb{Z}$ because $h(0)\iota(a) = h \cdot \iota(a)$, and $h(i)\iota(a) = 0$ for $i > 0$.

Let $S(\hat{\mathbf{h}}^-)_+ = (\iota(1) \otimes S(\hat{\mathbf{h}}^-)) \cap (V_L)_+$. The following lemma is a direct consequence of (5.11):

LEMMA 5.1. For $a, b \in \hat{L}$, we have

$$Y(\iota(a), z)\iota(b) = z^{\langle \bar{a}, \bar{b} \rangle} \exp\left(\sum_{n=1}^{\infty} \frac{\bar{a}(-n)}{n} z^n\right) \iota(ab). \quad (5.17)$$

In particular,

$$\iota(a)_n \iota(b) = 0 \quad \text{for } n \geq -\langle \bar{a}, \bar{b} \rangle, \quad (5.18)$$

$$\iota(a)_n \iota(b) = \iota(ab) \quad \text{for } n = -1 - \langle \bar{a}, \bar{b} \rangle, \quad (5.19)$$

$$\iota(a)_n \iota(b) \in S(\hat{\mathbf{h}}^-)_+ \iota(ab) \quad \text{for } n < -1 - \langle \bar{a}, \bar{b} \rangle. \quad (5.20)$$

Now we shall compute $C_1(V_L)$ by finding a spanning set of $C_1(V_L)$ among the elements of type (5.13). Let v , a , and k be as in (5.13). Then we have the following cases:

If $k \geq 2$, then

$$v = h^{i_1}(-n_1)(\iota(a) \otimes h^{i_2}(-n_2) \cdots h^{i_k}(-n_k)) \in C_1(V_L); \quad (5.21)$$

if $k = 1$ and $\bar{a} \neq 0$, then

$$v = h^{i_1}(-n_1) \iota(a) \in C_1(V_L); \quad (5.22)$$

if $k = 1$ and $\bar{a} = 0$, but $n = n_1 \geq 2$, then by (5.15)

$$v = 1 \otimes h(-n) = \frac{1}{n-1} L(-1)(1 \otimes h(-n+1)) \in C_1(V_L); \quad (5.23)$$

if $k = 1$, $\bar{a} = 0$, $n = 1$, we get $v = 1 \otimes h(-1) = h \in \mathbf{h}$.

Now, consider the case when $k = 0$. If $a = bc$ for some $b, c \in \hat{L}$ such that $\bar{b} \neq 0$, $\bar{c} \neq 0$, and $n = \langle \bar{b}, \bar{c} \rangle \geq 0$, then by (5.19) $\iota(a) = \iota(b)_{-n-1} \iota(c) \in C_1(V_L)$. Therefore, for α in L , $\iota(e_\alpha) \in C_1(V_L)$ if there exists $\beta \in L$ such that $\beta \neq 0$, α and $\langle \alpha - \beta, \beta \rangle \geq 0$ because

$$\iota(e_{\alpha-\beta})_{-1-\langle \alpha-\beta, \beta \rangle} \iota(e_\beta) = \epsilon(\alpha - \beta, \beta) \iota(e_\alpha).$$

This leads us to the following definition:

DEFINITION 5.2. We define

$$\Phi(L) = \{0 \neq \alpha \in L \mid \langle \alpha - \beta, \beta \rangle < 0 \text{ for every } \beta \neq 0, \alpha\}. \quad (5.24)$$

With the notion of $\Phi(L)$, we have

$$\iota(e_\alpha) \in C_1(V_L) \quad \text{for } \alpha \in L - (\Phi(L) \cup \{0\}). \quad (5.25)$$

Let K be the subspace of $C_1(V_L)$ linearly spanned by elements

$$v = \iota(a) \otimes h^{i_1}(-n_1) h^{i_2}(-n_2) \cdots h^{i_k}(-n_k),$$

where either (1) $k \geq 2$, or (2) $k = 1$ and $\bar{a} \neq 0$, or (3) $k = 1$ and $\bar{a} = 0$ but $n_1 \geq 2$, or (4) $v = \iota(a)$ for $a \in \hat{L}$, $\bar{a} \notin \Phi(L) \cup \{0\}$. Then we have:

PROPOSITION 5.3. $C_1(V_L) = K$.

Proof. We must prove that $C_1(V_L) \subseteq K$. From (5.15) we have

$$L(-1)V_L \subseteq K. \quad (5.26)$$

Then it suffices to show that $u_{-1}v \in K$ for every $u, v \in (V_L)_+$. We will show that $u_{-r}v \in K$ for every $r \geq 1$.

Observe that

$$h^{i_1}(-m_1) \cdots h^{i_\ell}(-m_\ell) \cdot V_L \subseteq K \quad \text{if } m_1 + \cdots + m_\ell \geq 2, \quad (5.27)$$

$$h^{i_1}(-m_1) \cdots h^{i_\ell}(-m_\ell) \cdot (V_L)_+ \subseteq K \quad \text{if } m_1 + \cdots + m_\ell \geq 1. \quad (5.28)$$

Claim I. If $r \geq 1$ and $a, b \in \hat{L}$ with $\bar{a} \neq 0, \bar{b} \neq 0$, then $\iota(a)_{-r}\iota(b) \in K$.

Let $a, b \in \hat{L}$ with $\bar{a} \neq 0, \bar{b} \neq 0$. Then (cf. (5.17))

$$\iota(a)_{-r}\iota(b) = \text{Res}_z z^{-r+\langle \bar{a}, \bar{b} \rangle} \exp\left(\sum_{n=1}^{\infty} \frac{\bar{a}(-n)}{n} z^n\right) \iota(ab). \quad (5.29)$$

If $\langle \bar{a}, \bar{b} \rangle \geq 0$, then $\bar{a} + \bar{b} \notin \Phi(L)$ and $\bar{a} + \bar{b} \neq 0$, so $\iota(ab) \in K$. Since by (5.28) $S(\hat{\mathbf{h}}^-)K \subseteq K$, we have $\iota(a)_{-r}\iota(b) \in K$.

If $\langle \bar{a}, \bar{b} \rangle < 0$, then it follows from (5.29) that $\iota(a)_{-r}\iota(b)$ is a linear combination of elements of the form

$$v = \iota(ab) \otimes \bar{a}(-n_1) \cdots \bar{a}(-n_k), \quad (5.30)$$

with $n_1 + \cdots + n_k = r - \langle \bar{a}, \bar{b} \rangle - 1 \geq 1$. If $\bar{a}\bar{b} = \bar{a} + \bar{b} \neq 0$, then by (5.28) each such v lies in K . If $\bar{a}\bar{b} = \bar{a} + \bar{b} = 0$, then $n_1 + \cdots + n_k = r + \langle \bar{a}, \bar{a} \rangle - 1 \geq 2$; hence by (5.27) each such v lies in K . Thus $\iota(a)_{-r}\iota(b) \in K$. The proof of Claim I is complete.

Claim II. $\iota(a)_{-r}(V_L)_+ \subseteq K$ for $r \geq 1$ and $a \in \hat{L}$ with $\bar{a} \neq 0$.

For $a \in \hat{L}$ with $\bar{a} \neq 0$, set

$$A_a = \{v \in (V_L)_+ \mid \iota(a)_{-r}v \in K \text{ for } r \geq 1\}.$$

By Claim I, $\iota(b) \in A_a$ for $b \in \hat{L}$ with $\bar{b} \neq 0$. Let $v \in A_a$, $h \in \mathbf{h}$, $s \geq 1$. Then by (5.16)

$$\iota(a)_{-r}h(-s)v = h(-s)\iota(a)_{-r}v - \langle \alpha, h \rangle \iota(a)_{-r-s}v \in K. \quad (5.31)$$

So $h(-s)v \in A_a$. Since $V_L = U(\hat{\mathbf{h}}^-)\mathbb{C}\{L\}$, we have $A_a = (V_L)_+$. This proves Claim II.

Claim III. $u_{-r}v \in K$ for any $u, v \in (V_L)_+$, $r \geq 1$.

Set

$$B = \{u \in (V_L)_+ \mid u_{-r}(V_L)_+ \subseteq K \text{ for } r \geq 1\}.$$

By (5.28) $\mathbf{h} \subseteq B$ and by Claim II, $\iota(a) \in B$ for $a \in \hat{L}$ with $\bar{a} \neq 0$. Let $u \in B$ and $h \in \mathbf{h}$ and $r, s \geq 1$. For $v \in (V_L)_+$, by the iterate formula (2.5), we get

$$(h(-r)u)_{-s}v = \sum_{i \geq 0} \binom{-r}{i} \left((-1)^i h_{-r-i} u_{-s+i} v - (-1)^{-r+i} u_{-r-s-i} h_i v \right). \quad (5.32)$$

Since $h_{-n}V_L \subseteq K$ for $n \geq 2$ (from (5.27)), we have $h_{-r-i}u_{-s+i}v \in K$ for $i > 0$. By assumption, we have $u_{-s}v \in K$; hence $h_{-r}u_{-s}v \in K$ (by (5.28)). Thus, for any $i \geq 0$,

$$h_{-r}u_{-s}v \in K.$$

For each $i \geq 0$, write $h_i v = \lambda_i \mathbf{1} + f^{(i)}$, where $\lambda_i \in \mathbb{C}$ and $f^{(i)} \in (V_L)_+$. By assumption, we have $u_{-r-s-i}f^{(i)} \in K$. Since $L(-1)V_L \subseteq K$, we have

$$u_{-r-s-i}(\lambda_i \mathbf{1}) = \frac{1}{r+s+i-1} \lambda_i L(-1)u_{-r-s-i+1} \mathbf{1} \in K.$$

Then $u_{-r-s-i}h_i v \in K$. Therefore $(h(-r)u)_{-s}v \subseteq K$. We have $h(-r)u \in B$. Again, since $V_L = S(\hat{\mathbf{h}}^-)\mathbb{C}\{L\}$, we get $B = (V_L)_+$. This proves Claim III and concludes the proof of Proposition 5.3. ■

COROLLARY 5.4. *Let $U = \mathbf{h} + \sum_{\alpha \in \Phi(L)} \mathbb{C}\iota(e_\alpha)$. Then $(V_L)_+ = U \oplus C_1(V_L)$. Consequently, U is a minimal generating subspace of weak PBW type for V_L .*

Proof. Choose a basis \mathcal{M} of $(V_L)_+$ out of elements (5.13) with $a = \iota(e_\alpha)$ for $\alpha \in L$. Clearly, $(V_L)_+ = U \oplus C_1(V_L)$ because the basis \mathcal{M} of $(V_L)_+$ is the disjoint union of two sets, one of them spanning U and the other spanning $K = C_1(V_L)$. ■

Remark 5.5. In [DLM4], in order to characterize the quotient space $V_L/C_2(V_L)$, which has a Poisson algebra structure, a subset of L bigger than $\Phi(L)$ was defined.

Next, we present some properties of $\Phi(L)$, which are shared by (finite) root systems (cf. [Hum]).

PROPOSITION 5.6. *The subset $\Phi(L)$ of a positive-definite even lattice L satisfies the following properties: (i) $\Phi(L)$ is finite and spans L over \mathbb{Z} ; (ii) if $\alpha \in \Phi(L)$, $n \in \mathbb{Z}$, then $n\alpha \in \Phi(L)$ if and only if $n = \pm 1$; (iii) $\Phi(L)$ is stable under the action of the automorphism group of the lattice L ; and (iv) $\langle \alpha, \beta \rangle < \langle \alpha, \alpha \rangle$ for $\alpha, \beta \in \Phi(L)$, $\alpha \neq \beta$.*

Proof. Let $n = \text{rank } L$ and let $\alpha_1, \dots, \alpha_n$ be n orthogonal elements of L , which are \mathbb{Q} -linearly independent in $\mathbb{Q} \otimes_{\mathbb{Z}} L$. Let $\alpha \in \Phi(L)$. If $\alpha \neq \pm \alpha_i$ for any i , the $\langle \alpha \mp \alpha_i, \pm \alpha_i \rangle < 0$ for every i . Thus

$$|\langle \alpha, \alpha_i \rangle| < \langle \alpha_i, \alpha_i \rangle \quad (5.33)$$

for every i . Then $\alpha = c_1\alpha_1 + \dots + c_n\alpha_n$ with $|c_i| < 1$ for $1 \leq i \leq n$. Therefore, $\Phi(L) - \{\pm \alpha_1, \dots, \pm \alpha_n\}$ is finite, so $\Phi(L)$ is finite.

Set $E = \mathbb{Z}\Phi(L) \subseteq L$. If $E \neq L$, let α be an element of $L - E$ with the smallest norm. Since $\alpha \notin \Phi(L)$, there is $\beta \in L$ such that $\beta \neq 0$, α and $\langle \alpha - \beta, \beta \rangle \geq 0$. Since

$$\langle \alpha, \alpha \rangle = \langle \alpha - \beta, \alpha - \beta \rangle + 2\langle \alpha - \beta, \beta \rangle + \langle \beta, \beta \rangle,$$

both $\alpha - \beta$ and β have norms smaller than the norm of α , so $\alpha - \beta$ and β lie in E . Thus $\alpha \in E$. This is a contradiction.

Let G be the symmetry group of the lattice L . It follows directly from the definition of $\Phi(L)$ that $G\Phi(L) = \Phi(L)$. Since -1 is an automorphism of the lattice L , $-\alpha \in \Phi(L)$ if $\alpha \in \Phi(L)$. For any $n \geq 2$, we have $\langle n\alpha - \alpha, \alpha \rangle > 0$. Thus $n\alpha \notin \Phi(L)$, and hence $-n\alpha \notin \Phi(L)$. Part (iv) follows directly from the definition. ■

The following lemma claims that nonzero elements of L with minimal norm are in $\Phi(L)$.

LEMMA 5.7. *Suppose that $\langle \alpha, \alpha \rangle \geq 2k$ for every $\alpha \in L - \{0\}$, where k is a fixed positive integer. Then $L_{2k} \subseteq \Phi(L)$, where*

$$L_{2k} = \{\alpha \in L | \langle \alpha, \alpha \rangle = 2k\}. \quad (5.34)$$

Proof. Suppose that $\alpha \in L_{2k}$ and $\beta \in L$. Then

$$2k = \langle \alpha, \alpha \rangle = \langle \alpha - \beta, \alpha - \beta \rangle + 2\langle \alpha - \beta, \beta \rangle + \langle \beta, \beta \rangle.$$

If $\beta \neq 0$, α , then $\langle \beta, \beta \rangle, \langle \alpha - \beta, \alpha - \beta \rangle \geq 2k$, so $2\langle \alpha - \beta, \beta \rangle \leq -2k < 0$. Thus $\alpha \in \Phi(L)$. ■

Next we shall find $\Phi(L)$ for some special L .

PROPOSITION 5.8. *Let Φ be a simply laced root system and let k be a positive integer. Set $L = \mathbb{Z}\sqrt{k}\Phi$. Then $\Phi(L) = L_{2k} = \sqrt{k}\Phi$.*

Proof. It follows directly from the definition that $\Phi(\sqrt{k}L) = \sqrt{k}\Phi(L)$ for any L . Thus it suffices to prove the proposition for $k = 1$. Since $\Phi = L_2$, we have $\Phi \subseteq \Phi(L)$ by Lemma 5.7.

Assume that $\Phi(L) \neq \Phi$, and consider any $\beta \in \Phi(L) - \Phi$. For $\alpha \in \Phi$, we have $\langle \beta \pm \alpha, \mp \alpha \rangle < 0$, so

$$-2 = -\langle \alpha, \alpha \rangle < \langle \beta, \alpha \rangle < \langle \alpha, \alpha \rangle = 2. \quad (5.35)$$

Let W be the Weyl group of Φ . Then $W\Phi(L) = \Phi(L)$ and $W(\Phi(L) - \Phi) = \Phi(L) - \Phi$. Choose a Weyl chamber so that $\Phi = \Phi_+ \cup \Phi_-$. Let $\mathbf{Q}_+ = \mathbb{Z}_+\Phi_+$. Using \mathbf{Q}_+ , we can define a partial order on L (cf. [Hum, K]). Let β be a maximal element of the orbit $W\beta$. Since, for $\alpha \in \Phi$,

$$r_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha \in W\beta,$$

where r_α is the reflection associated to α , we have $\langle \beta, \alpha \rangle \geq 0$ for $\alpha \in \Phi_+$. Then by (5.35)

$$\langle \beta, \alpha \rangle = 0 \text{ or } 1 \quad \text{for every } \alpha \in \Phi_+. \quad (5.36)$$

By Lemma 5.1, (5.35), and (5.36) we obtain

$$\iota(e_\gamma)_n \iota(e_\beta) = 0 \quad \text{for } \gamma \in \Phi, n \geq 1, \quad (5.37)$$

$$\iota(e_\alpha)_0 \iota(e_\beta) = 0 \quad \text{for } \alpha \in \Phi_+. \quad (5.38)$$

It is known (cf. [FLM]) that the weight 1 subspace $(V_L)_{(1)}$ carries the structure of a Lie algebra \mathfrak{g} and that V_L is a module for the affine Lie algebra $\hat{\mathfrak{g}}$. Since L is a root lattice, V_L is an irreducible $\hat{\mathfrak{g}}$ -module (cf. [FLM, Proposition 7.2.7]); hence $\mathbf{1}$ is the only highest weight vector up to scalar multiple. However, (5.37) and (5.38) imply that $\iota(e_\beta)$ is a highest weight vector with $\text{wt } \iota(e_\beta) = \frac{1}{2}\langle \beta, \beta \rangle > 0$ in V_L for the affine Lie algebra $\hat{\mathfrak{g}}$ (cf. [K]). This is a contradiction. ■

Remark 5.9. In proving the result above, one can instead finish the argument by checking case by case that there does not exist $0 \neq \beta \in L$ such that $\langle \beta, \alpha \rangle = 0$ or 1 for every $\alpha \in \Phi_+$. Here we chose to use the $\hat{\mathfrak{g}}$ -module structure on V_L to avoid checking cases.

Remark 5.10. It is known [B] that $V/L(-1)V$ has a Lie algebra structure with $[\bar{u}, \bar{v}] = \overline{u_0v}$ for $u, v \in V$, where \bar{a} denotes the coset $a + L(-1)V$ for $a \in V$. Since

$$u_0v_{-1}w = v_{-1}u_0w + (u_0v)_{-1}w$$

for $u, v, w \in V$, $C_1(V)/L(-1)V$ is an ideal. Thus $V/C_1(V)$ has a natural Lie algebra structure. Let U be a graded vector subspace of V such that $V_+ = U \oplus C_1(V)$. Since there is a natural linear bijection from U to $V/C_1(V)$, U carries a Lie algebra structure. In particular, if $V = V_L$, then $U = \mathbf{h} + \sum_{\alpha \in \Phi(L)} \mathbb{C} \iota(e_\alpha)$ has a Lie algebra structure, which is given as follows: For $h \in \mathbf{h}$, $\alpha, \beta \in \Phi(L)$,

$$[h, \iota(e_\alpha)] = \langle \alpha, h \rangle \iota(e_\alpha), \quad (5.39)$$

$$[\iota(e_\alpha), \iota(e_\beta)] = \epsilon(\alpha, \beta) \iota(e_{\alpha+\beta}) \quad \text{if } \langle \alpha, \beta \rangle = -1, \alpha + \beta \in \Phi(L), \quad (5.40)$$

$$[\iota(e_\alpha), \iota(e_\beta)] = 0 \quad \text{if } \langle \alpha, \beta \rangle = -1, \alpha + \beta \notin \Phi(L), \quad (5.41)$$

$$[\iota(e_\alpha), \iota(e_\beta)] = 0 \quad \text{if } \langle \alpha, \beta \rangle \neq -1, \alpha + \beta \neq 0, \quad (5.42)$$

$$[\iota(e_\alpha), \iota(e_{-\alpha})] = \alpha \quad \text{if } \langle \alpha, \alpha \rangle = 2, \quad (5.43)$$

$$[\iota(e_\alpha), \iota(e_{-\alpha})] = 0 \quad \text{if } \langle \alpha, \alpha \rangle > 2. \quad (5.44)$$

This calculation uses Lemma 5.1 and the fact that $K = C_1(V_L)$, i.e., Proposition 5.3. If $L = \mathbb{Z}L_2$, then $U = (V_L)_{(1)}$ is a semisimple Lie algebra [FLM, Chap. 6]. However, if $L \neq \mathbb{Z}L_2$, by calculating the kernel of the Killing form, one sees that U has a nontrivial radical, which contains e_α for every $\alpha \in \Phi(L) - L_2$.

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